PROBLEMS FOR TEAM CONTEST

ANSWER ALL QUESTIONS

Problem 1. The following statement *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \le n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \le H, \quad 1 \le i \le m,\ 1 \le j \le n.$$

(i) Show that for $K \ge 0$ there are at most $(2K+1)^n$ vectors $\mathbf{x} \in \mathbb{Z}^n$ with

$$\|\mathbf{x}\|_\infty \le K,$$

where $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$.
(ii) Apply (i) and Dirichlet's pigeon hole principle to prove that there is an integer **non-zero** vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \le (2nH)^{n-1}.$$

Problem 2. Let $u(x), a(x)$ and $f(x)$ be smooth functions on $[0,1]$.
(i) Determine the order of accuracy of the following approximation

$$\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]\bigg|_{x=x_i} \simeq \frac{(a_{i+1}+a_i)(u_{i+1}-u_i)-(a_i+a_{i-1})(u_i-u_{i-1})}{2h^2}$$

where $h = \frac{1}{m+1}$ is the mesh size, $x_i = ih$, $a_i = a(x_i)$, and $u_i = u(x_i)$ for $i = 1, \cdots, m$ such that $x_{m+1} = 1$.
(ii) For given functions $a(x) > 0$ and $f(x)$, one determines the function u that solves the following second order ordinary differential equation

$$u - \frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x)$$

with boundary conditions $u(0) = 0$ and $u(1) = 0$. Apply the discretization given in (i) and let $f_i = f(x_i)$. Denote the linear system that one has to solve by $A\mathbf{u} = \mathbf{f}$ where $A \in \mathbb{R}^{m\times m}$ and $\mathbf{u}, \mathbf{f} \in \mathbb{R}^m$. If Gauss-Seidel method is used to solve this linear system, show that the iterative method converges for any initial guess.

Problem 3. *Maximal entropy principle.* Consider probability distributions on a discrete random variable X taking on possible values of $x_1, x_2, \ldots, x_n$. Denote the probability $\Pr(X = x_i) = p_i$, $i = 1, \ldots, n$, and recall that its Shannon entropy S is

$$S = -\sum_{i=1}^{n} p_i \log p_i$$

Now suppose we have some knowledge of $p_1, \ldots, p_n$, specified in terms of its expectation values E_j with respects to k known functions $f_j(\cdot)$ of the random variable X

$$\sum_{i=1}^{n} p_i f_j(x_i) = E_j, \qquad j = 1, 2, \ldots, k < n.$$

(i) Show that the probability distribution $p = [p_1, \ldots, p_n]$ that maximized the entropy S has the form of an exponential family:

$$p_i = \frac{e^{\sum_{j=1}^{k} \lambda_j f_j(x_i)}}{Z}$$

where λ_j are all constants, and Z is the normalization constant given by

$$Z = \sum_{i=1}^{n} e^{\sum_{j=1}^{k} \lambda_j f_j(x_i)}$$

(ii) Show that the constants λ_j are related to E_j by

$$E_j = \frac{\partial \log Z}{\partial \lambda_j}.$$